## 1. Introduction

*"Prediction is difficult, especially of the future."*

Niels Bohr[1]

This thesis consists of two parts in which neither part deals explicitly with prediction. However, the difficulties of prediction are both the origin of and the common link in this thesis, even though each part tackles the problem from a different point of view. That prediction is a problem for financial markets is revealed by the very existence of insurance arrangements for those markets - if the future was not associated with uncertainty, there would be no risks and insurance markets would not exist.

The first part of this thesis, which consists of Papers [i]-[iii], deals with a possible explanation of why prediction is difficult, namely, that observed time series may be chaotic. More specifically, this part of the thesis proposes a method to answer the following question: are observed time series, *e.g.* exchange rate series, generated by a chaotic dynamical system? If so, the predictive ability of the system is strongly limited, especially for long-run predictions.

The second part of this thesis, which consists of Paper [iv], deals with a simple solution to the prediction problem, namely, that there is predictive power in certain trading rules, *i.e.* in technical analysis. Two questions, which are also more important than the possibility of chaos, are in focus in this part of the thesis. They are as follows: are technical analysts in the foreign exchange market destabilizing actors in the economy? If so, are there any stability conditions that prevent the economy from "exploding"?

The remainder of this Introduction and Summary of Papers is organized as follows. Section 2 contains a review on how to detect chaotic dynamics in observed time series. Short summaries of Papers [i]-[iii] are also included. Section 3 contains a short introduction to technical analysis in general and to theories of endogenous speculative bubbles in particular. A summary of Paper [iv] is also included in this

[1] This quotation can be found in Peitgen *et al.* (1992).

section. Concluding remarks can be found in Section 4.

## 2. Detecting chaotic dynamics in observed time series

Deterministic chaos is about the erratic or stochastic behavior of solutions to deterministic equations of motion, *i.e.* deterministic dynamical systems, and has received much attention over recent years in many diverse fields including economics and finance. Broadly speaking, a dynamical system is said to be deterministic when it comprises no random variables. In contrast, the observable behavior, *i.e.* the observed time series, of a dynamical system is said to be stochastic when the transition of the system from one state to another can only be given a probabilistic description.

One important implication of chaos is the limited predictability of the dynamics. This is because a chaotic dynamical system has a property of "sensitive dependence on initial conditions": any two solution paths with arbitrarily close but not equal initial conditions will diverge at exponential rates. Globally, however, the solution paths remain within a bounded set if the system is dissipative, *i.e.* if the system has internal "friction". Thus, the future development of the dynamics is essentially unpredictable even though the underlying equations of motion are deterministic. Because predictability is important, especially in economics and finance, the question of whether observed time series, *e.g.* exchange rate series, are generated by a chaotic dynamical system or not is of special importance.[2]

### 2.1. Reconstruction of the dynamics

*The first step in resolving the question of whether the observed dynamics are chaotic or not, is taken by reconstruction of the phase space or state space for the dynamics.* Specifically, under the conditions stated below, a map exists between the original phase space and the reconstructed phase space that is an embedding, *i.e.* the map is

[2] Abarbanel (1996) and Abarbanel *et al.* (1993) provide reviews on the detection of chaotic dynamics in observed time series. However, these reviews do not contain any distributional theory which provides a framework for statistical inference.

smooth, performs a one-to-one coordinate transformation and has a smooth inverse. This means that the map preserves topological information about the dynamical system under the mapping, *e.g.* the dimension, the entropy and the Lyapunov exponents which are to be defined below.

Phase space reconstruction was introduced as a tool in dynamical systems theory by Packard *et al.* (1980), David Ruelle and Takens (1981) independently.[3] The method was demonstrated numerically by Packard *et al.* (1980) and was formally proven by Takens (1981). The most widely used method of phase space reconstruction is delay coordinates. According to this method, the past and the future of an observed scalar time series contain information about unobserved state variables that can be used to define a state at the present time. Other methods of phase space reconstruction are derivative coordinates of which Packard *et al.* (1980) is an example, and principal value decomposition which was originally proposed by Broomhead and King (1986). Both methods use the information in delay coordinates as a starting point for a further transformation to a new coordinate system (Casdagli *et al.*, 1991).

### 2.1.1. Embeddings

Let $f$ be a smooth dynamical system defined on an $n$-dimensional compact manifold $M$:

$$s(t) = f^t(s(0)), \tag{2.1}$$

where $s(t)$ is the state of the system at time $t$. The map $f^t$ thus takes an initial state $s(0)$ to a state $s(t)$ where the time variable $t$ can either be continuous or discrete. Associate with the dynamical system in eq. (2.1) an observer function $h : M \to \mathbf{R}$

[3] The idea of phase space reconstruction was not published by David Ruelle but is recorded in Packard *et al.* (1980).

which generates the data points in the observed scalar time series:

$$\begin{cases} x(t) = h(s(t)) + \gamma\varepsilon(t) \\ \varepsilon(t) \sim IID(0,1) \end{cases}, \tag{2.2}$$

where $\gamma$ is the noise level and $\varepsilon(t)$ is the measurement error. More precisely, assume that the observer function, sampled at time intervals $\tau_s$, generates the $N$-point time series

$$\{x_1, ..., x_N\}, \tag{2.3}$$

where $x_t \equiv x(t)$. Note that observational noise is present in the observed time series.[4] For the moment, however, it is assumed that the observed time series is noise-free, *i.e.* $\gamma = 0$ in eq. (2.2).

As mentioned above, the past and the future of an observed scalar time series contain information about unobserved state variables that can be used to define a state at the present time. Specifically, let

$$\underline{x_t} = \{x_{t-Jm_p}, ..., x_t, ..., x_{t+Jm_f}\} \tag{2.4}$$

be the reconstructed state at time $t$ where $J \in \mathbf{Z}^+$ is the reconstruction delay and, $m_p$ and $m_f$ are the number of data points in the time series taken from the past and the future, respectively. The lag time is $\tau = J\tau_s$, the window length is $\tau_w = m\tau$ and the dimension of the delay vector in eq. (2.4), *i.e.* the embedding dimension, is $m = m_p + m_f + 1$. Moreover, the number of states on the reconstructed trajectory in the reconstructed phase space are $N_{ts} = N - (m-1)J$. Thus, the reconstructed trajectory is $\underline{X} = \{\underline{x_1}, ..., \underline{x_{ts}}\}^T$ where $\underline{X}$ is an $N_{ts} \times m$ matrix and the superscript $T$ denotes transpose.

For example, assume that the observed time series consists of ten data points, *i.e.* $N = 10$ in eq. (2.3). Further assume that the reconstruction delay is equal to

[4] Casdagli *et al.* (1991) deal with the problem of phase space reconstruction in the presence of observational noise.

two, *i.e.* $J = 2$ in eq. (2.4), and that $m_p = 0$ and $m_f = 3$ in eq. (2.4). This means that the reconstructed trajectory can be written as

$$\underline{X} = \{\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4\}^T = \begin{pmatrix} x_1 & x_3 & x_5 & x_7 \\ x_2 & x_4 & x_6 & x_8 \\ x_3 & x_5 & x_7 & x_9 \\ x_4 & x_6 & x_8 & x_{10} \end{pmatrix}.$$

Takens (1981) proved that the delay reconstruction map

$$\Phi(s) = \{h(f^{\tau m_f}(s)), ..., h(f^0(s)), ..., h(f^{-\tau m_p}(s))\}, \tag{2.5}$$

which maps the original $n$-dimensional state $s$ onto $m$-dimensional delay vectors, is an embedding if $m > 2n$.[5] If $\Phi$ is an embedding then a smooth function $g$ is induced on the reconstructed phase space:

$$g^t(\underline{x}) = \Phi \circ f^t \circ \Phi^{-1}(\underline{x}), \tag{2.6}$$

where $\circ$ is the operator for the composition of mappings. The reconstructed phase space can thus be used to estimate $g$ and because $g$ is topologically conjugate to the true but unknown dynamical system $f$ in eq. (2.1), *i.e.* they are "equivalent", $g$ can be used to estimate relevant statistical properties of the dynamics, *e.g.* the Lyapunov exponents.

An intuitive explanation of Takens' (1981) embedding criterion, *i.e.* $m > 2n$, is as follows: Assume that $M_1 \subset M_r$ and $M_2 \subset M_r$ are two subspaces of dimension $n_1$ and $n_2$, respectively, where $M_r$ is an $m$-dimensional manifold representing the reconstructed phase space. In general, the two subspaces intersect in a subspace of dimension $n_1 + n_2 - m$. Thus, if this expression is negative, there is no intersection of the two subspaces. Of greater interest, the intersection of two subspaces of the same dimension, or the self-intersection of an $n$-dimensional set with itself fails to

[5] An early paper on embeddings is Whitney (1936). See also a related paper by Mañé (1981).

occur when $n + n - m < 0$, *i.e.* when $m > 2n$.

Sauer *et al.* (1991) contains several generalizations of Takens' (1981) embedding theorems. For example, for a chaotic attractor $A \subset \mathbf{R}^n$, almost every delay reconstruction map $\Phi : \mathbf{R}^n \to \mathbf{R}^m$ is an embedding on smooth manifolds contained within $A$ provided that $m$ is twice as large as the dimension of $A$. Broadly speaking, an attractor is a compact set $A$ with the property that there is a neighborhood of $A$ such that for almost every initial condition, the limit set of the trajectory defined by the dynamical system, *i.e.* once transients have disappeared, is $A$. These initial conditions are the basin of attraction of the attractor. The distinction between asymptotic and transient behavior is possible if it is assumed that the dynamical system is dissipative, *i.e.* if volumes in phase space shrink over time.

### 2.1.2. Choosing the proper reconstruction parameters

**Embedding dimension** In practice, there are two problems with delay coordinates as a method of phase space reconstruction. The first problem, which is common to all reconstruction methods, is that the necessary dimension of the reconstructed phase space, *i.e.* the minimum embedding dimension, is not known since the dimension of the original phase space is also unknown. This problem may, however, be solved indirectly by utilizing a generic property of a faithful reconstruction, *i.e.* an embedding, namely, that the attractor in the original phase space is completely "unfolded", or contains no self-intersections, in the reconstructed phase space. In other words, if the embedding dimension is too low, the attractor is not completely unfolded which means that distant points on the attractor in the original phase space are close points in the reconstructed phase space. Moreover, if the attractor is completely unfolded it is not necessary to view the attractor in a phase space with a higher embedding dimension. Thus, from a theoretical point of view, it does not matter whether the attractor is unfolded in an embedding dimension which is too high.

From a practical point of view, however, there are several reasons to determine the minimum embedding dimension $m_m$. First, the number of points on the reconstructed attractor can be too few to obtain reliable estimates of, for example, the Lyapunov exponents. Second, the computational "cost" rises exponentially with the embedding dimension because all mathematical computations take place in $\mathbf{R}^m$. Third, in the presence of noise, the unnecessary dimensions of the phase space, *i.e.* $m - m_m$, are not populated by new information about the attractor because this information has already been captured in a smaller embedding dimension. In summary, it is desirable to determine the minimum embedding dimension.

Examples of methods developed for this purpose are false nearest neighbors, saturation of invariants on the attractor and true vector fields. The method of false nearest neighbors is based on the aforementioned property of a faithful reconstruction, namely, that when choosing an embedding dimension which is too low, distant points on the attractor in the original phase space are close points in the reconstructed phase space, *i.e.* false nearest neighbors exist. By increasing the embedding dimension, the attractor is completely unfolded when there are no false nearest neighbors (Kennel *et al.*, 1992).

The second method, saturation of invariants on the attractor, is based on the observation that when the attractor is completely unfolded, any invariant on the attractor, *e.g.* the dimension of the reconstructed attractor, is independent of the embedding dimension. If, however, the attractor is not completely unfolded in the reconstructed phase space, these invariants depend on the embedding dimension. Thus, by increasing the embedding dimension, the attractor is completely unfolded when the value of the specific invariant on the attractor stops changing (Grassberger and Procaccia, 1983).

The third method, true vector fields, is based on another property of a faithful reconstruction, namely, that the vector field associated with the vector function $f^t$ is unambiguous when the attractor is completely unfolded. In other words, the tangents to the evolution of the vector function are smoothly and uniquely given throughout the phase space. Thus, if the embedding dimension is too low, the vector field in some neighborhoods of the attractor is not unique because the directional vectors, *i.e.* the tangents, in that neighborhood point in different directions. Thus, by increasing the embedding dimension, the attractor is completely unfolded when the directional vectors in each neighborhood point in the same direction. Kaplan and Glass (1992) specifically test whether the distribution of directions in a neighborhood of the attractor is consistent with being generated by a deterministic system.

When observational noise is present in the observed scalar time series, *i.e.* $\gamma \neq 0$ in eq. (2.2), it is not an easy task to determine the minimum embedding dimension, especially when the noise to signal ratio is high. For example, if the method of saturation of invariants on the attractor is utilized, the specific invariant may never stop changing.

**Reconstruction delay** A second problem with delay coordinates, which is specific to this reconstruction method, is that the reconstruction delay, *i.e.* $J$ in eq. (2.4), must be chosen. If the reconstruction delay is too small, each coordinate is almost the same and this results in a reconstructed attractor that is compressed along the identity line, or the main diagonal, of the phase space. If, however, the reconstruction delay is too large, successive delay coordinates may become causally unrelated and the reconstructed attractor no longer represents the true dynamics. These problems are the problems of redundance and irrelevance, respectively. It is thus desirable to determine the proper reconstruction delay. It should be noted that Takens' (1981) embedding theorems did not address problems about the proper reconstruction delay because if the observed scalar time series is noise-free, *i.e.* $\gamma = 0$ in eq. (2.2), almost every reconstruction delay is suitable.

Examples of methods to determine the proper reconstruction delay are average mutual information and reconstruction expansion from the identity line. The latter method is a geometrical method that focuses on the redundance error in a proposed reconstruction of the attractor. This method, proposed by Rosenstein *et al.* (1994), is based on the observation that the expansion of the reconstructed attractor from the identity line is best quantified by measuring the average displacement of the delay vectors, *i.e.* $\underline{x}_t$ in eq. (2.4), from their original locations on the identity line. These authors suggest that the proper reconstruction delay has been determined when the first maximum of the average displacement examined as a function of the reconstruction delay, occurs. This method thus neglects the irrelevance error in a proposed reconstruction of the attractor, because one typically cannot measure the irrelevance error.

The former method, average mutual information, focuses on the chaotic dynamical system as a producer of information. Because the resolution in the reconstructed phase space is, in practice, finite, points which are too close together cannot be distinguished. However, because close points separate exponentially when the system is chaotic, they can be distinguished if enough time has elapsed. The fact that these points were close but not equal is revealed as time evolves. This means that if the reconstruction delay is too small, the chaotic dynamical system has not yet explored enough of its phase space to produce new information about that phase space. Fraser and Swinney (1986) make use of the so-called average mutual information function when they determine the proper reconstruction delay. The average mutual information function can be considered as a generalization of the autocorrelation function where the latter provides a measure of the linear dependence, on the average over all observations, between measurements at a certain lag. Specifically, the average mutual information function gives the average amount of information about $x_{t+J}$ given $x_t$. Fraser and Swinney (1986) suggest that when the first minimum of the average mutual information function occurs, the proper reconstruction delay is determined.

### 2.2. Ergodic quantities

*The second step in resolving the question of whether the observed dynamics are chaotic or not, is taken by estimation of relevant statistical properties of the time evolution, i.e. so-called ergodic quantities.* A basic virtue of the ergodic theory of dynamical systems is that it allows one to consider only the asymptotic properties of a dynamical system and thus neglect the transients.[6] The focus, however, is on invariant probability measures rather than on attractors, which is the case within the geometric approach to dynamical systems.

With reference to the dynamical system in eq. (2.1) and the associated observer function in eq. (2.2), letting $\gamma \neq 0$, *i.e.* observational noise is present, an experimental time average of the observed time series may exist:

$$\lim_{t' \to \infty} \frac{1}{t'} \int_0^{t'} h(s(t))dt. \tag{2.7}$$

This time average in eq. (2.7) produces an invariant probability measure $\rho$ on the attractor $A \subset M$ which describes how frequently various parts of $A$ are visited by the trajectory, *i.e.* the time average of the observable $h(s(t))$ is equal to its space average $\rho(h(s))$. The probability measure $\rho$ is invariant under the time evolution if $\rho(\varphi \circ f^t) = \rho(\varphi)$ for every continuous function $\varphi$.

An invariant probability measure $\rho$ is ergodic if it does not have a non-trivial convex decomposition, *i.e.* $\rho = c\rho_1 + (1-c)\rho_2$ where $c \in (0,1)$, $\rho_1$ and $\rho_2$ are again invariant probability measures, and $\rho_1 \neq \rho_2$. Because, in general, an invariant probability measure $\rho$ can be uniquely represented as a superposition of ergodic measures, it is natural to assume $\rho$ to be ergodic. If so, the ergodic theorem asserts that

$$\rho(\varphi) \equiv \int \varphi(s)\rho(ds) = \lim_{t' \to \infty} \frac{1}{t'} \int_0^{t'} \varphi(s(t))dt$$

for every continuous function $\varphi$ and for almost all initial conditions $s(0)$ with respect to that measure $\rho$. This means that it makes sense to discuss the invariant statistical properties of the time evolution, *i.e.* ergodic quantities, like the Lyapunov exponents, the Kolmogorov (1958)-Sinaĭ (1959) entropy and various definitions of the dimension of an attractor.

In general, however, it is exceptional that an attractor carries only one ergodic

[6]Introductions to the ergodic theory of dynamical systems are found in Eckmann and Ruelle (1985) and Ruelle (1989).

measure. For example, a so-called strange attractor, *i.e.* a chaotic attractor, typically carries uncountably many distinct ergodic measures. Normally, when observational noise is present in the observed time series, there is only one stationary measure $\rho\gamma$.[7] It may be the case that this so-called physical measure tends to a specific ergodic measure when $\gamma \to 0$. Hereafter, however, it is assumed that any invariant probability measure carried by an attractor is ergodic.

The Lyapunov exponents for the dynamical system in eq. (2.1) with respect to the measure $\rho$ measure the average exponential divergence or convergence of nearby but not equal initial conditions. Because the dynamical system is defined on an $n$-dimensional manifold, there are $n$ Lyapunov exponents ranked from the largest to the smallest. A positive Lyapunov exponent measures the average exponential divergence of nearby initial conditions whereas a negative exponent measures the average exponential convergence. Thus, if we refer to the property of "sensitive dependence on initial conditions", a positive Lyapunov exponent is present when the dynamics are chaotic.

The Kolmogorov (1958)-Sinaĭ (1959) entropy of the measure $\rho$ uses the fact that dynamical systems produce information and measures the average rate at which this information is produced. In the deterministic case, this means that the entropy of a non-chaotic dynamical system producing the measure $\rho$ is zero because on average, only chaotic dynamical systems produce new information about the attractor generated by the system. In Pesin (1977), it is stated and proven that the Kolmogorov (1958)-Sinaĭ (1959) entropy of a dynamical system is equal to the sum of the positive Lyapunov exponents whenever a physical measure $\rho$ is present.

As mentioned above, there are a number of definitions of the dimension of an attractor (Farmer *et al.*, 1983). The dimension $D_A$ of a set, *e.g.* an attractor, is the amount of information needed to specify its points accurately, and is related to how hyper-volumes $v$ scale as a function of a length parameter $l$, *i.e.* $v \propto l^{D_A}$.

[7] Chan and Tong (1994) prove that, under appropriate conditions, an embedded deterministic dynamical system which admits an attractor can give rise to an ergodic stochastic system. This observation justifies the stochastic setup in eq. (2.2).

For example, areas vary with the square of the length of the side and volumes vary with the cube. For fractal sets the dimension typically takes a non-integer value. The dimension of a dynamical system is important because it reveals the number of variables that are sufficient to mimic the systems behavior.

### 2.2.1. Lyapunov exponents

**Definition** Specifically, let $s(0)$ and $s(0)'$ be two initial conditions that are close to each other, *i.e.* $\|s(0) - s(0)'\| < \varepsilon$ where $\|.\|$ is any norm and $\varepsilon > 0$ is small. Assume for simplicity that the time evolution is discrete. After time $N$,

$$\begin{aligned} s(N) - s(N)' &= f^N(s(0)) - f^N(s(0)') \\ &\approx Df^N(s(0))(s(0) - s(0)') \\ &= Df(s(N-1)) \times Df(s(N-2)) \times ... \times Df(s(0))(s(0) - s(0)'), \end{aligned}$$

where $Df$ is the Jacobian matrix. Moreover,

$$\begin{aligned} \|s(N) - s(N)'\|^2 &= (s(0) - s(0)')^T (Df^N(s))^T Df^N(s)(s(0) - s(0)') \\ &= ...(Df^N(s))^T Df^N(s)... \end{aligned}$$

The multiplicative ergodic theorem by Oseledec (1968) states that if we form the so-called Oseledec matrix

$$OSL(s, N) = ((Df^N(s))^T Df^N(s))^{\frac{1}{2N}},$$

then the limit $\lim_{N\to\infty} OSL(s, N)$, exists and is independent for $\rho$-almost all $s$ in the basin of attraction of the attractor to which the trajectory defined by the dynamical system belongs. The logarithm of the eigenvalues of $\lim_{N\to\infty} OSL(s, N)$, which is an orthogonal matrix, are denoted by $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and are the global Lyapunov exponents for the dynamical system with respect to the ergodic measure $\rho$.

The global Lyapunov exponents give a measure of the average growth or decay of infinitesimal perturbations to a trajectory over a long time. Closely related to the global Lyapunov exponents, hereafter called the Lyapunov exponents, are the local or finite time Lyapunov exponents (Abarbanel *et al.*, 1991). The local Lyapunov exponents $\lambda_i(s, N)$ measure how rapidly infinitesimal perturbations to a trajectory at state $s$ grow or shrink in $N$ time steps away from the time of the perturbation. Often, these exponents vary significantly with the state $s$ on the attractor, especially when $N$ is small, which means that the variation of predictability over the attractor may be large. Of course, $\lim_{N\to\infty} \lambda_i(s, N) = \lambda_i$.

If the largest Lyapunov exponent is positive, *i.e.* $\lambda_1 > 0$, the system is chaotic and has the aforementioned property of "sensitive dependence on initial conditions". If the system is dissipative then the sum of the Lyapunov exponents is negative, *i.e.* $\sum_{i=1}^{n} \lambda_i < 0$, and the solution paths remain within a bounded set.

**Estimation** Since the seminal paper by Takens (1981), extensive research has been carried out to develop methods for estimating the Lyapunov exponents from an observed scalar time series. The first method was presented by Wolf *et al.* (1985) and thereafter several methods have been proposed.[8] These methods either estimate the largest Lyapunov exponent, the non-negative portion of the Lyapunov exponent spectrum or the entire Lyapunov exponent spectrum.

Most methods are so-called Jacobian methods. Recall that if the delay reconstruction map $\Phi$ in eq. (2.5) is an embedding, then the function $g$ in eq. (2.6) is induced on the reconstructed phase space. Since the Lyapunov exponents are computed from $Df^N$, it is important to estimate $Dg^N$. To simplify the exposition, assume that the reconstruction delay is equal to one and that the reconstruction is predictive in the sense that no data points are taken from the future, *i.e.* $J = 1$ and

[8] Examples are Abarbanel *et al.* (1992), Barna and Tsuda (1993), Briggs (1990), Brown *et al.* (1991), Dechert and Gençay (1992), Eckmann *et al.* (1986), Ellner *et al.* (1991), Gençay and Dechert (1992), Kurths and Herzel (1987), McCaffrey *et al.* (1992), Nychka *et al.* (1992), Rosenstein *et al.* (1993), Sano and Sawada (1985), Stoop and Parisi (1991) and Zeng *et al.* (1991). Zeng *et al.* (1992) is a review that compares various methods for the estimation of the Lyapunov exponents from an observed scalar time series.

$m_f = 0$ in eq. (2.4). This means that the function $g$ may be written as

$$g : \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-m+1} \end{bmatrix} \rightarrow \begin{bmatrix} x_{t+1} = g_1(x_t, x_{t-1}, ..., x_{t-m+1}) \\ x_t \\ \vdots \\ x_{t-m+2} \end{bmatrix}.$$

The Jacobian matrix $Dg$ at the reconstructed state $\underline{x}_t$ is

$$(Dg)_{\underline{x}_t} = \begin{pmatrix} \frac{\partial g_1}{\partial x_t} & \frac{\partial g_1}{\partial x_{t-1}} & \frac{\partial g_1}{\partial x_{t-2}} & & \frac{\partial g_1}{\partial x_{t-m+2}} & \frac{\partial g_1}{\partial x_{t-m+1}} \\ 1 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & 0 \end{pmatrix}.$$

Since

$$(Dg^N)_{\underline{x}_0} = (Dg)_{\underline{x}_{N-1}} \times (Dg)_{\underline{x}_{N-2}} \times ... \times (Dg)_{\underline{x}_0},$$

the Lyapunov exponents can be estimated by estimating the Jacobian matrixes along the reconstructed trajectory. For example, Dechert and Gençay (1992) and Gençay and Dechert (1992) consistently estimate the entire Lyapunov exponent spectrum whereas McCaffrey *et al.* (1992) and Nychka *et al.* (1992) consistently estimate the largest Lyapunov exponent.

There is, however, a problem with the Jacobian method when estimating the entire Lyapunov exponent spectrum. This is because the method is applied to the full $m$-dimensional reconstructed phase space and not just to the $n$-dimensional manifold which is the image of the delay reconstruction map $\Phi$ in eq. (2.5). As a consequence, $m$ Lyapunov exponents are estimated whereupon $n$ are true Lyapunov exponents and $m - n$ are so-called spurious Lyapunov exponents. Even worse, spurious Lyapunov exponents that are larger than the largest true Lyapunov exponent may be obtained (Dechert and Gençay, 1996). The true Lyapunov exponents can, however, be identified because they change their signs upon time reversal whereas the spurious Lyapunov exponents do not (Parlitz, 1992). It goes without saying that the method does not work when the dynamics fail to be one-to-one.

A simple non-Jacobian method that estimates only the largest Lyapunov exponent was proposed by Rosenstein *et al.* (1993). This method, which is based on previous work by Eckmann *et al.* (1986) and Sano and Sawada (1985), is utilized in Papers [i]-[iii] and is as follows: Given a reconstruction of the attractor, *i.e.* a reconstructed phase space, search for the nearest neighbor of each state on the trajectory:

$$d_t(0) = \min_{\underline{x}_{t'} \neq \underline{x}_t} \left\| \underline{x}_t - \underline{x}_{t'} \right\|,$$

where $d_t(0)$, $\underline{x}_t$ and $\underline{x}_{t'}$ are the distances between the $t$th state and its nearest neighbor, the reference state and the nearest neighbor, respectively. The $t$th pair of nearest neighbors then diverge at a rate approximated by the largest Lyapunov exponent:

$$d_t(i) \approx d_t(0) \exp(\lambda_1 i), \tag{2.8}$$

where $i$ is the number of steps following the nearest neighbors. Taking the logarithm of both sides of eq. (2.8) gives

$$\log d_t(i) \approx \log d_t(0) + \lambda_1 i,$$

which represents a set of approximately parallel lines each with a slope approximately proportional to $\lambda_1$. The largest Lyapunov exponent is then estimated using a least-squares fit with a constant to the average line defined by $\langle \log d_t(i) \rangle$ where $\langle . \rangle$ denotes the average over all values of $t$.

## 2.3. Testing for chaotic dynamics in observed time series

*The third and final step in resolving the question of whether the observed dynamics are chaotic or not, is taken by applying statistical inference to the largest Lyapunov* *exponent.* For example, observational noise may be present in the observed time series, *i.e.* $\gamma \neq 0$ in eq. (2.2). Moreover, the time series may contain dynamic noise, *i.e.* the unknown system may be a stochastic dynamical system. Therefore, a distributional theory which provides a framework for statistical inference is needed.

Recently, Bailey (1996) and Whang and Linton (1997) have derived analytically the asymptotic distributions of estimators of the Lyapunov exponents for stochastic time series. Bailey (1996) deals with local Lyapunov exponents whereas Whang and Linton (1997) deal with global exponents.

### 2.3.1. Statistical framework

The statistical framework proposed in Paper [ii] is based on a previous work by Gençay (1996a), and utilizes a moving blocks bootstrap procedure to test for the presence of a positive Lyapunov exponent in an observed stochastic time series.[9] One weakness with the proposed framework is that the dynamics are non-chaotic under the null hypothesis.

The main idea with bootstrapping is to use the available observations to design a sort of Monte Carlo experiment in which the observations themselves are used to approximate the distribution of the error terms or other random quantities (Efron, 1979). Künsch (1989) and Liu and Singh (1992) extended the idea of bootstrapping to the case where the observations form a stationary sequence.[10]

Specifically, consider a sequence $\{X_1, ..., X_N\}$ of weakly dependent stationary random variables. According to Künsch (1989) and Liu and Singh (1992), the distribution of certain estimators can be consistently constructed by blockwise bootstrap.[11] Therefore, let $B_t$ denote a moving block of $b$ consecutive observations, *i.e.* $B_t = \{X_t, ..., X_{t+b-1}\}$. If $k$ satisfy $bk \sim N$, then resample with replacement, $k$ blocks from the sequence $\{B_1, ..., B_{N-b+1}\}$. Denote the resulting sampled blocks by

[9] See also Lai and Chen (1995) for a related paper that was shown to me by Dejian Lai after Paper [ii] was published in Physica D.

[10] See Sjöstedt (1997) for a review of various resampling techniques for both stationary and non-stationary sequences of random variables.

[11] Roughly speaking, with "certain" estimators one typically refer to smooth functions of means.

$\xi_1, ..., \xi_k$ and concatenate these blocks into one vector $\{\xi_1, ..., \xi_k\}$ which constitutes the bootstrap sample.

Let $\theta$ be the unknown parameter of interest, *e.g.* the largest Lyapunov exponent, $\hat{\theta}$ its considered estimator and $\tilde{\theta}$ the statistic computed from the bootstrap sample. By obtaining a large number of bootstrap values $\sqrt{N}(\tilde{\theta} - \hat{\theta})$, one can estimate the distribution of $\sqrt{N}(\hat{\theta} - \theta)$. More precisely, the bootstrap values form an empirical distribution which can be utilized in statistical hypothesis testing.

The idea behind the test schemes below is simply to let the states on the reconstructed trajectory to be the moving blocks.

**Test schemes** The null hypothesis $H_0$ and the alternative hypothesis $H_1$ are formulated as

$$H_0 : \lambda^{\max} = 0 \qquad vs. \qquad H_1 : \lambda^{\max} > 0,$$

where $\lambda^{\max}$ is the unknown parameter, *i.e.* the largest Lyapunov exponent. Under the null hypothesis, the dynamics are non-chaotic.

Two test schemes are presented; one is general and the other is simplified. The general test scheme allows a reconstruction delay that is different from one, *i.e.* $J \neq 1$ in eq. (2.4), whereas the simplified test scheme restricts the reconstruction delay to one, *i.e.* $J = 1$ in eq. (2.4). The latter test scheme is presented because in practice, most reconstructions of the dynamics use a reconstruction delay that is equal to one.

**The general test scheme consists of the following steps:**

(i) Estimate the proper reconstruction delay $\hat{J}_p$.

(ii) Reconstruct the attractor from the observed scalar $N$-point time series where $m = m'$ and $J = \hat{J}_p$, and estimate the largest Lyapunov exponent $\hat{\lambda}_1$. The embedding dimension $m'$ should satisfy Takens' (1981) embedding criterion, *i.e.* $m' > 2D_A$ where $D_A$ is the dimension of the attractor.

(iii) Reconstruct the attractor from the observed scalar $N$-point time series where $m = b$ and $J = 1$. The selected block size satisfies $b = (m' - 1)\hat{J}_p + 1$ and should also satisfy Takens' (1981) embedding criterion, *i.e.* $b > (2D_A - 1)\hat{J}_p + 1$. Denote the states on the reconstructed trajectory, *i.e.* the delay vectors, by $T_1, ..., T_M$ where the number of states are $N_{ts} = N - b + 1$.

(iv) Resample with replacement, $k$ blocks from the sequence $\{T_1, ..., T_M\}$ where $k = N \bmod b$. Denote the resulting sampled blocks by $\xi_1, ..., \xi_k$.

(v) Utilize the projection $P(\xi_t) = \{x_t, x_{t+\hat{J}_p}, ..., x_{t+(m'-1)\hat{J}_p}\}$ where $P : \mathbf{R}^{(m'-1)\hat{J}_p+1} \to \mathbf{R}^{m'}$. The sequence $\{P(\xi_1), ..., P(\xi_k)\}$ constitutes the bootstrap sample.

(vi) Estimate the bootstrap value of the largest Lyapunov exponent $\tilde{\lambda}_1$ from the bootstrap sample and calculate $\tilde{\lambda}_1 - \hat{\lambda}_1$.

(vii) Repeat steps (iv)-(vi) a large number of times in order to construct an empirical distribution for $\tilde{\lambda}_1 - \hat{\lambda}_1$.

(viii) Construct a one-sided confidence interval, *e.g.* a 95% confidence interval, by calculating the critical value as $\hat{\lambda}_1 - q(95\%)$, following from $E_q\{\Pr\{(\hat{\lambda}_1 - \lambda_1) \leq q(95\%)\}\} = 0.95$, where $q(95\%)$ is the quantile for the distribution in step (vii).

(ix) If $\hat{\lambda}_1 - q(95\%) > 0$, then the null hypothesis is rejected.

**The simplified test scheme consists of the following steps:**

(i) Reconstruct the attractor from the observed scalar $N$-point time series where $m = b$ and $J = 1$, and estimate the largest Lyapunov exponent $\hat{\lambda}_1$. The embedding dimension $b$, *i.e.* the selected block size, should satisfy Takens' (1981) embedding criterion, *i.e.* $b > 2D_A$ where $D_A$ is the dimension of the attractor. Denote the states on the reconstructed trajectory, *i.e.* the delay vectors, by $T_1, ..., T_M$ where the number of states are $N_{ts} = N - b + 1$.

(ii) Resample with replacement, $k$ blocks from the sequence $\{T_1, ..., T_M\}$ where $k = N \bmod b$. Denote the resulting sampled blocks by $\xi_1, ..., \xi_k$. The sequence $\{\xi_1, ..., \xi_k\}$ constitutes the bootstrap sample.

(iii) Estimate the bootstrap value of the largest Lyapunov exponent $\tilde{\lambda}_1$ from the bootstrap sample and calculate $\tilde{\lambda}_1 - \hat{\lambda}_1$.

(iv) Repeat steps (ii)-(iii) a large number of times in order to construct an empirical distribution for $\tilde{\lambda}_1 - \hat{\lambda}_1$.

(v) Construct a one-sided confidence interval, *e.g.* a 95% confidence interval, by calculating the critical value as $\hat{\lambda}_1 - q(95\%)$, following from $E_q\{\Pr\{(\hat{\lambda}_1 - \lambda_1) \leq q(95\%)\}\} = 0.95$, where $q(95\%)$ is the quantile for the distribution in step (iv).

(vi) If $\hat{\lambda}_1 - q(95\%) > 0$, then the null hypothesis is rejected.

**Differential equations or discrete time maps as the sources of the observed dynamics?** The proposed framework above can also be utilized to discriminate between differential equations, *i.e.* a continuous dynamical system, and discrete time maps, *i.e.* a discrete dynamical system, as the sources of observed dynamics. If the dynamics are generated by differential equations, *i.e.* a flow, then at least one of the Lyapunov exponents is equal to zero (Abarbanel, 1996, and Eckmann and Ruelle, 1985). The reason is that if we choose to make a perturbation to a trajectory in the same direction as the trajectory is going then that perturbation will simply move us along the same trajectory on which we started. Thus, there is no divergence between the "new" trajectory and the "old" trajectory, *i.e.* $\lambda_i = 0$ for that particular direction. In the case of finite time maps, there are no flow directions. Thus, if one of the Lyapunov exponents is zero we can be confident that we have a flow.

In particular, a two-sided test statistic can be designed in the proposed framework:

$$H_0 : \lambda = 0 \qquad vs. \qquad H_1 : \lambda \neq 0,$$

where $\lambda$ is the unknown parameter. Under the null hypothesis, the dynamics are generated by differential equations. Thus, if the null hypothesis is rejected for each Lyapunov exponent in the entire Lyapunov exponent spectrum, then the dynamics cannot be generated by differential equations. This information may be useful when modelling the observed dynamics.

### 2.4. Short summaries of Papers [i]-[iii]

#### 2.4.1. Paper [i]

This paper presents estimates for the largest Lyapunov exponents for some selected exchange rate series, thus revealing the local stability properties of the dynamical system generating these exchange rate series. Estimates of the correlation dimension are also given which means that the number of variables sufficient to mimic a system's behavior are determined. The correlation dimension, originally proposed by Grassberger and Procaccia (1983), provides a measure of the spatial correlation of states on an attractor generated by a dynamical system. The exchange rate series that were examined are the Swedish Krona versus Deutsche Mark, ECU, U.S. Dollar and Yen exchange rates. Daily data from January, 1986 to August, 1995 were used. Each exchange rate series includes 2 409 data points which, in the field of economics, represent rather long time series.

The estimated values of the largest Lyapunov exponents were positive in all exchange rate series suggesting that the behavior of these time series is chaotic. The correlation dimension estimates revealed a low-dimensional dynamical system in the case of the Swedish Krona-ECU and possibly also in the case of the Swedish Krona-U.S. Dollar. In the other exchange rate series, the correlation dimension estimates continued to increase as the embedding dimension was increased. Thus, the attractor was never completely unfolded in the reconstructed phase space which is an indication of a high noise to signal ratio.

One of the shortcomings of this paper is the absence of a distributional theory that provides a framework for statistical inference of the estimates.

#### 2.4.2. Paper [ii]

This paper presents a statistical framework based on a blockwise bootstrap procedure that tests for the presence of a positive Lyapunov exponent in an observed stochastic time series. Because a positive Lyapunov exponent is an operational definition of deterministic chaos if the dynamical system generating the time series is dissipative, the proposed framework is a test for deterministic chaos. Specifically, the null hypothesis $H_0$ and the alternative hypothesis $H_1$ are formulated as

$$H_0 : \lambda^{\max} = 0 \qquad vs. \qquad H_1 : \lambda^{\max} > 0,$$

where $\lambda^{\max}$ is the unknown parameter, *i.e.* the largest Lyapunov exponent. As can be seen, the dynamics are non-chaotic under the null hypothesis which is a weakness of this framework.

The proposed framework was tested on the Hénon (1976) map which is described by the following map:

$$\begin{cases} x_{t+1} = y_t + 1 - a x_t^2 \\ y_{t+1} = b x_t \end{cases} . \tag{2.9}$$

If $a = 1.4$ and $b = 0.3$ in eq. (2.9), then the sequence of points obtained by iteration of the mapping either diverges to infinity or tends to a strange attractor. When the initial point is $(x_0, y_0) = (0, 0)$, the sequence of points $\{(x_0, y_0), (x_1, y_1), ...\}$ tend to a strange attractor.

Test samples were constructed by the following "observer" map:

$$\begin{cases} x_t^o = x_t + \gamma\varepsilon_t \\ \varepsilon_t \sim NID(0,1) \end{cases} ,$$

where $\gamma$ is the noise level and $\varepsilon_t$ is the measurement error. Thus, the "observed" time series, *i.e.* the test samples, was $\{x_{1001}^o, ..., x_{1000+N}^o\}$ where $N = 200$ and $N = 1000$. The noise levels were $\gamma = 0.05$ and $\gamma = 0.25$, block sizes of 2, 4, 6, 8 and 10 were used in each test and 400 bootstrap values were calculated. The null hypothesis was rejected in all cases for a significance level of 0.025. This is in accordance with the true largest Lyapunov exponent, which in this case is $\lambda_1^{true} = 0.408$.

### 2.4.3. Paper [iii]

Can nominal exchange rates be characterized by deterministic chaos? In order to answer this question, the statistical framework proposed in Paper [ii] was used. The Swedish Krona versus Deutsche Mark, ECU, U.S. Dollar and Yen exchange rate series were examined using daily data from May 17, 1991 to August 31, 1995. Because the Swedish Krona was pegged against the ECU between May 17, 1991 and November 19, 1992, the exchange rate series were divided into two parts where the first part includes exchange rates from May 17, 1991 to November 19, 1992 and the second part includes exchange rates from November 20, 1992 to August 31, 1995. This was done to separate the dynamics from the fixed and flexible exchange rate periods.

In most cases, the null hypothesis that the exchange rate series cannot be characterized by deterministic chaos was rejected. Therefore, one answer to the question posed may be yes. This conclusion must, however, be treated with caution for several reasons. First, the results in this paper are not in accordance with the results in Jonsson (1997), who used daily data for the Swedish Krona versus the U.S. Dollar from November 20, 1992 to December 30, 1994. Since Jonsson (1997) used a method proposed by Nychka *et al.* (1992) that yields consistent estimates of the largest Lyapunov exponent, our results suggest that the method of estimation proposed by Rosenstein *et al.* (1993) may be upwardly biased. The rate of convergence for Nychka's *et al.* (1992) method is not, however, known and the sample size used in Jonsson (1997) is small. Further investigation of these potential sources of difference is clearly warranted.

## 3. Technical analysis in the foreign exchange market

The ability to predict the foreign exchange rate is important. For example, a currency exposure would like to know whether a hedge is desirable or not, and a currency trader would like to know which positions will be profitable. The demand for accurate forecasts is therefore high.

In order to predict the future exchange rate, the following question is relevant: what factors determine the current and thus the future exchange rate? Because the exchange rate is an important price in the economy, much research has been carried out to answer this question. The results of these efforts have been discouraging. Meese and Rogoff (1983), for example, demonstrated that the predictive abilities of various exchange rate models were very low. Moreover, to quote Frankel and Froot (1990): *"... the proportion of exchange rate movements that can be explained even after the fact, using contemporaneous macroeconomic variables, is disturbingly low"*.

Frankel and Froot (1986, 1990), therefore, called for a new theory called endogenous speculative bubbles. Within this new research area, the fact that various agents in financial markets, such as the foreign exchange markets, use technical analysis in their trade is taken into account. A degree of irrationality is thus introduced into the models. However, to quote Shleifer and Summers (1990): *"It is absolutely not true that introducing a degree of irrationality of some investors into models of financial markets "eliminates all discipline and can explain anything""*.

### 3.1. Technical analysis

Technical analysis assumes that there are patterns in market prices, *e.g.* in exchange rates, that will recur in the future and that these patterns can be used for predictive purposes. This means that technical analysis directly contradicts the hypothesis of efficient markets. In other words, technical analysts, or chartists, use the past history of, for example, the exchange rate to detect patterns which they extrapolate into the future utilizing technical trading rules. These trading rules are heuristic forecasting methods which chartists claim give them an edge in forecasting the movements of market prices, *e.g.* the movements of exchange rates.

Papers that present evidence that there is predictive power contained in some of the trading rules used by chartists are Brock *et al.* (1992), Gençay (1996b), LeBaron (1996), Levich and Thomas (1993), Sweeney (1986) and Taylor (1992). Specifically, it appears that the predictive ability is greatest for the foreign exchange markets and, of interest from a practical point of view, that the magnitude of trading profitability makes up for the costs of trading in these markets. It thus seems that Michael Jensen, the founding editor of Journal of Financial Economics, was too hasty when he stated that *"... there is no other proposition in economics which has more solid empirical evidence supporting it than the Efficient Market Hypothesis"* (Jensen, 1978).

### 3.1.1. Moving averages

One common trading rule used by chartists is the moving average rule (Taylor and Allen, 1992). In its simplest form, a buy signal is generated when the market price is above a (long-period) moving average of that market price:

$$s_t > \frac{1}{n}\sum_{i=0}^{n-1} s_{t-i}, \tag{3.1}$$

where $s$ is the market price and $n$ is the number of days in the moving average. Otherwise, a sell signal is generated.[12]

In practice, eq. (3.1) is replaced by more advanced trading rules.[13] For example,

[12] A caution: If $s$ is the market price of the exchange rate, which can be defined as the amount of domestic currency one has to pay for one unit of foreign currency, a buy signal is generated when the market price, *i.e.* the exchange rate, is below a moving average of that market price. This is because a rising exchange rate indicates that the currency is losing value.

[13] Assuming that the time evolution is continuous, Paper [iv] shows that an exponential moving average of the past history of market prices:

$$MA(t) = \int_{-\infty}^{t} \upsilon(\tau)s(\tau)d\tau,$$

where

$$\int_{-\infty}^{t} \upsilon(\tau)d\tau = \int_{-\infty}^{t} \omega \exp(\omega(\tau - t))d\tau = 1,$$

where $\omega > 0$, is equal to

$$MA(t) = s(t) + \sum_{i=1}^{\infty}(-1)^i \frac{1}{\omega^i}\frac{d^i s(t)}{dt^i}.$$

Thus, this exponential moving average of the past history of market prices can be written as the market price today plus an infinite series of time derivatives of the market price today. This observation may shed light on why delay coordinates, discussed in Section 2 above, work as a tool of phase space reconstruction, *i.e.* that the past and the future of an observed scalar time series contain information about unobserved state variables that can be used to define a state the market price today may be replaced by a short-period moving average. Thus, when a short-period moving average is above a long-period moving average, a buy signal is generated. The most popular moving average rules used in technical analysis are the 1-50, 1-150, 1-200, 2-200 and the 5-200 rules (Gençay, 1996b). The 5-200 rule, for example, means that the short-period moving average includes 5 days and that the long-period moving average includes 200 days. Moreover, to avoid so-called whiplash signals, which are generated when the short-period and the long-period moving averages are similar, a band between these moving averages may be introduced.

#### 3.1.2. Momentum lines

Another, but less common, trading rule used by chartists is the momentum line rule (Taylor and Allen, 1992). According to this trading rule, a buy signal is generated when the current rate of change begins to increase relative to the rate of change $n$ days ago. A sell signal is otherwise generated.[14] In its simplest form, $n = 0$, which means that a buy signal is generated when the current rate of change begins to increase.

### 3.2. Endogenous speculative bubbles

A rather new research area in the theory of exchange rates are theories of endogenous speculative bubbles (Frankel and Rose, 1995). A number of researchers have deviated from the rational expectations paradigm within this research area. Theories of endogenous speculative bubbles differ from theories of rational speculative bubbles because the latter models are based on rational expectations.[15] Dominguez (1986) and Ito (1990, 1994), for example, use survey data on exchange rate expectations at

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at the present time. Because derivative coordinates are in some sense more intuitive, the above relationship may demystify delay coordinates. Finally, Sauer *et al.* (1991) prove the more general case of phase space reconstructions which use moving averages of delay coordinates.

[14] Note the caution in footnote 12.

[15] Rosser (1997) reviews a variety of issues related to speculative bubbles, *e.g.* rational speculative bubbles.

various horizons to question the hypothesis of rational expectations in the foreign exchange market.

Specifically, the models within theories of endogenous speculative bubbles start from the proposition that the forecasts of the market participants are drawn from competing views, *i.e.* different kinds of actors are introduced into the models. This proposition is justified by Cutler *et al.* (1991), Ito (1990) and Taylor and Allen (1992) who provide empirical evidence for the importance of considering heterogeneous agents in exchange rate theory.

In a seminal paper by Frankel and Froot (1986), three kinds of actors were introduced; fundamentalists, portfolio managers and the aforementioned chartists. These three actors behave differently when forming their expectations about the future development of the exchange rate because they have different information sets. Fundamentalists have macroeconomic fundamentals in their information set, *e.g.* the exchange rate in long-run equilibrium, and thus base their expectations according to a model that consists of macroeconomic fundamentals only, *e.g.* the Dornbusch (1976) overshooting model. Chartists, on the other hand, have information sets containing only the time series of the exchange rate itself.

Portfolio managers, the only actors who actually buy and sell foreign currencies, form their expectations about the future development of the exchange rate as a weighted average of the expectations of chartists and fundamentalists:

$$E(\frac{ds}{dt}) = E_c(\frac{ds}{dt})\omega + E_f(\frac{ds}{dt})(1-\omega), \tag{3.2}$$

where $\frac{ds}{dt}$ denotes the rate of depreciation of the domestic currency, $E_c(.)$ and $E_f(.)$ denote expectations of chartists and fundamentalists, respectively, and $E(.)$, which is a weighted average of these expectations where $\omega$ determines the weights, denotes expectations of portfolio managers. Thus, each of these three actors behave in a "rational manner", but they behave differently because of their different information sets.

For example, the portfolio managers in Frankel and Froot (1986, 1990) update the weights over time according to whether the chartists or the fundamentalists have recently been doing the better forecasting. De Grauwe *et al.* (1993) provide another example of a model with heterogeneous agents. They showed that the interaction of chartists and fundamentalists in the foreign exchange market can give rise to a chaotic behavior of the exchange rate.

### 3.3. Summary of Paper [iv]

The theoretical contribution of this paper is found in the way in which the portfolio managers weight the expectations of chartists and fundamentalists. This consists of explicitly modelling the empirical observation that the relative importance of technical versus fundamental analysis in the foreign exchange market depends on the forecasting horizon. For shorter forecasting horizons, more weight is placed on technical analysis, while the opposite is true for longer forecasting horizons (Taylor and Allen, 1992). Furthermore, it is assumed that the forecasting horizon depends inversely on domestic inflation rate, *i.e.* the forecasting horizon is infinite if domestic prices are "stable", and myopic if domestic prices are "unstable". Hence, it is assumed that macroeconomic fundamental analysis dominates if domestic inflation rate is low, while technical analysis dominates if domestic inflation rate is high. Technical analysis also dominates if domestic deflation rate is high.

Thus, referring to eq. (3.2) above, the expectations for the portfolio managers can be written as

$$E(\frac{ds}{dt}) = E_c(\frac{ds}{dt})\exp(-\frac{1}{(\frac{dp}{dt})^2}) + E_f(\frac{ds}{dt})(1-\exp(-\frac{1}{(\frac{dp}{dt})^2})),$$

where $\frac{dp}{dt}$ is the domestic inflation rate.[16]

The theoretical framework in this paper is the Dornbusch (1976) overshooting model. The main assumptions of this model can be summarized as follows: (1) the economy is permanently fully employed, which implies that fluctuations in demands

[16] In order to avoid division by zero, $\varepsilon > 0$ may be introduced into the denominator of the fraction.

for goods only result in price movements, not in output movements; (2) goods prices are sticky in the short-run, *i.e.* they need time to react to and absorb fluctuations in goods demand; and (3) the domestic interest rate and the exchange rate are perfectly flexible, which implies that the monetary sector is in permanent equilibrium. The stickiness of goods prices is the reason for the overshooting phenomenon, *i.e.* the exchange rate overshoots its long-run equilibrium in the short-run because it restricts prices from making their required contribution to overall adjustment.

The Dornbusch (1976) overshooting model is simple and has a number of shortcomings, *e.g.* economic policy is treated exogenously and the economy is represented in an *ad hoc* fashion without an explicit microeconomic foundation. Despite these shortcomings, the Dornbusch (1976) overshooting model is used as the theoretical framework in this paper for two reasons. First, this model is suitable because nominal exchange rates are volatile (Frankel and Rose, 1995). Second, for technical analysis to exist, the foreign exchange market must be inefficient. The assumption that goods prices are assumed to be sticky in the short-run means that inefficiency is a characterizing feature of this model.

The contributions in this paper consist of four models. In these models, the fundamentalists have either adaptive expectations or perfect foresight, and the chartists use either moving averages or momentum lines to form their expectations. The combinations of expectation formations for the chartists and the fundamentalists are given in Table 3.1.[17]

Table 3.1: Expectation formations

| | Expectation formations | |
|---|---|---|
| Model | Chartists | Fundamentalists |
| (iii) | Moving averages | Adaptive expectations |
| (iv) | Momentum lines | Adaptive expectations |
| (v) | Moving averages | Perfect foresight |
| (vi) | Momentum lines | Perfect foresight |

[17]Models (i)-(ii) are excluded in Table 3.1. In these models, there are no chartists. The fundamentalists have adaptive expectations in model (i) and perfect foresight in model (ii).

Three questions are in focus in the formal analysis of models (iii)-(vi). First: are these models globally stable or is saddle path stability the characterizing feature? Second: what are the stability conditions for these models? Third: is the overshooting phenomenon robust with respect to the presence of chartists in these models?

*Are models (iii)-(vi) globally stable or is saddle path stability the characterizing feature?* All models are characterized by saddle path stability although global stability is also possible in model (iii). In short, the more flexible the goods prices, the more likely it is that the model is globally stable.

*What are the stability conditions for models (iii)-(vi)?* When saddle path stability is the characterizing feature of model (iii), the stability condition is that the economy must continuously be in equilibrium. This is also the stability condition for models (iv)-(vi). Because the monetary sector is assumed to be in permanent equilibrium, this stability condition means that the real sector, *i.e.* the goods market, must also be in equilibrium, which is the case when the domestic inflation rate is zero. This implies that the portfolio managers exclude the chartists when forming their expectations about the future development of the exchange rate, which means that chartists have no influence on the foreign exchange market.

The stability condition that the economy must continuously be in equilibrium can also be interpreted as suggesting that actual and expected inflation rates must be equal. As a consequence, if actual and expected inflation rates differ, the exchange rate will reach "infinity". Because the expected inflation rate is not a part of models (iii)-(vi), this conjecture forms a basis for further research.

*Is the overshooting phenomenon robust with respect to the presence of chartists in models (iii)-(vi)?* Yes, if the models are continuously in equilibrium as is the case in three of the models. The exception is model (iii) where global stability is possible. In this model, the exchange rate may even move in the "wrong direction" after a monetary disturbance, *i.e.* the currency loses value after a contractionary monetary policy, if the domestic inflation rate is modest. However, if the domestic inflation rate is low, the level of overshooting of the exchange rate is greater than when the Dornbusch (1976) overshooting model holds.

In summary, an economic policy that stabilizes the inflation rate at a low level may destabilize the exchange rate through overshooting. This, however, is the "price" one has to pay to prevent the economy from "exploding". Though, the recommended policy of a low inflation rate should not be interpreted in favor of a policy that give priority to a fully employed economy, because the latter, *i.e.* a fully employed economy, has been an assumption in all models.

## 4. Concluding remarks

Does chaos really matter? The answer to this question depends on who you are asking. For technical analysts, the answer is clearly no, even if patterns in market prices, such as exchange rates, never recur in exactly the same manner. Trading rules still have predictive power for technical analysts. Moreover, the prediction problem associated with chaos in, for example, foreign exchange markets with target zones, *i.e.* fixed exchange rate regimes with bands, may be negligible if the bands are small. For example, if the band is $\pm 1\%$ around some target rate, the prediction error can never be larger than 2%.

Chaos still has several important implications that must be considered in economics. For example, the prediction problem associated with chaos cannot be entirely neglected because the aforementioned example of target zones with small bands is an exception. In most cases, the bands are either large or they are absent, *i.e.* flexible exchange rates regimes are adopted. This means that nonlinear dynamics in general and chaotic dynamics in particular in the foreign exchange market have important implications for modelling purposes, *i.e.* for economic theory.

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